Ising model on the scale-free network with a Cayley-tree-like structure

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We derive an exact expression for the magnetization and the zero-field susceptibility of the Ising model on a random graph with degree distribution $P(k)\propto k^{-\gamma}$ and with a boundary consisting of leaves, that is, vertices whose degree is 1. The system has no magnetization at any finite temperature, and the susceptibility diverges below a certain temperature T_s depending on the exponent γ. In particular, T_s reaches infinity for $\gamma\leqslant 4$. These results are completely different from those of the case having no boundary, indicating the nontrivial roles of the leaves in the networks.

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I. INTRODUCTION

Recently, complex networks have been studied as models to describe topologically complex real-world systems [1,2]. So-called scale-free (SF) networks have been found in social systems, in protein interaction networks, in the internet, and in the worldwide web [3]. In a SF network the degree distribution $P(k)$, where degree k is the number of edges connected to a vertex, has a power-law decay $P(k)\propto k^{-\gamma}$. In real networks, the exponent γ is usually in the range $2<\gamma<3$ [4]. Various processes taking place on SF networks, e.g., network failure and spread of infections, have been studied to demonstrate that the behaviors on SF networks are far from those on periodic lattices due to the existence of *hubs*, that is, vertices having extremely high degree. Among them, the cooperative behaviors and dynamics of the interacting systems on such networks are of great interest [5–21]. So far, the Ising model on SF networks has been investigated by both analytical [5–11] and numerical methods [12–17] to show that the critical behavior strongly depends on the exponent γ.

In analyzing networks, it has been frequently assumed that the cyclic paths can be ignored and the structure is regarded as treelike. On a treelike structure their cooperative behavior critically depends on the number of *leaves*, that is, vertices whose degree is 1. For example, the Ising model on the Cayley tree and that on the Bethe lattice show different critical behavior. The ferromagnetic Ising model on the Bethe lattice has a transition temperature T_c [22] while that on the Cayley tree has no magnetization at any finite temperature [23]. The aim of the present work is to investigate this difference between the following two treelike structures on SF networks: (i) a Cayley-tree-like structure where the depth is finite and the tree is bounded by leaves, and (ii) a Bethe-lattice-like structure where the depth is infinite and there is no boundary. Dorogovtsev *et al.* [8] showed that the ferromagnetic Ising model on a SF network with the Bethe-lattice-like structure remains in the ferromagnetic phase at any finite temperature for $\gamma\leqslant 3$, while a phase transition exists at a finite temperature T_c for $\gamma>3$, and its critical exponents vary depending on the exponent γ. On the other hand, an analysis of a SF network with the Cayley-tree-like structure is still missing.

In this paper, we demonstrate the effect of leaf spins on a SF network. We show that the cooperative behaviors of the ferromagnetic Ising model on a SF network with a Cayley-tree-like structure differ entirely from those with a Bethe-lattice-like structure. We generalize the method by Stošić *et al.* [24] to derive the exact representations for the magnetization and the zero-field susceptibility. We show that the system has no magnetization at any finite temperature even if the exponent γ is small, and the susceptibility diverges below a certain temperature T_s which depends on γ. In particular, T_s reaches infinity and the susceptibility is divergent for any finite temperature $\gamma\leqslant 4$.

II. THE CONSTRUCTION OF THE SF CAYLEY TREE

The SF network with a Cayley-tree-like structure can be derived from the configuration model [25]. The configuration model allows one to sample graphs with a given degree sequence which will tend to the degree distribution $P(k)\propto k^{-\gamma}$ $(k\geqslant 3)$ for large N. Correlation between degrees of vertices in such a graph is absent. Now we choose a vertex randomly as the *root*, select its first nearest neighbors, and say that these vertices are on the first shell. We repeat the process to create further the rth shell by adding the rth nearest neighbors connecting to each vertex on the $(r-1)$th shell (Fig. 1).

Now we cut the branches emerging from the vertices on the nth shell to extract a subgraph, where the vertices on the nth shell reduce to the leaves. The resulting subgraph is a tree, since the whole graph has a locally treelike structure. We call this graph a SF Cayley tree with radius n. If we take the radius n as infinity and ignore the leaf spins on the boundary, the SF Cayley tree reduces to the model Dorogovtsev *et al.* [8] analyzed, which we call a SF Bethe lattice. We append the indices to the vertices as follows. We denote (i) the root by $\vec{i}_0=(i_0=1)$, (ii) the vertices on the first shell by $\vec{i}_1=(\vec{i}_0,i_1)=(i_0,i_1)$ where the number of index i_1 is equal to the number of branches emerging from the root $\vec{i}_0$, (iii) the vertices on the second shell by $\vec{i}_2=(\vec{i}_1,i_2)=(i_0,i_1,i_2)$ where the number of index i_2 with given $\vec{i}_1$ is equal to the number

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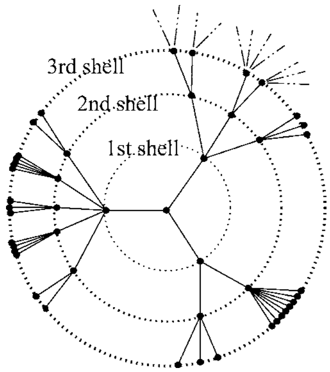

FIG. 1. Example of a SF Cayley tree with radius $n=3$. We cut branches emerging from the vertices on the third shell (dotted lines); thus the degree of the vertices on the third shell is 1.

of branches emerging from the vertex $\vec{i}_1$, and so on. The index of each vertex on the mth shell ($1 \leqslant m \leqslant n$) is given by $\vec{i}_m = (i_0, i_1, i_2, \ldots, i_m)$. The average number of vertices on the mth shell is $[\langle k(k-1)\rangle / \langle k \rangle]^{m-1} \langle k \rangle$. Here $\langle \cdots \rangle$ means the average with $P(k)$, e.g., $\langle k \rangle = \Sigma k P(k)$.

Now we consider a ferromagnetic Ising spin system on a SF Cayley tree with radius n. The Hamiltonian is

$$H = -J \sum_{\langle ij \rangle} S_i S_j - h \sum_i S_i, \tag{1}$$

where J (>0) is the ferromagnetic interaction, h is the external magnetic field, and S_i ($=\pm 1$) is the Ising spin variable on the vertex i. The first sum is over all edges of the graph, and the second one is over all vertices.

III. THE DERIVATION OF THE RECURSION RELATIONS

Stošić *et al.* derived the exact expression for the magnetization and the zero-field susceptibility of the Ising model on the regular Cayley tree with arbitrary radius n by using the recursion relations between the partition functions with two consecutive radii [24]. We generalize their method to obtain the magnetization and the susceptibility on the SF Cayley tree with radius n. At first we consider the subtree whose root is the vertex $\vec{i}_m$. Note that, when the root vertex is $\vec{i}_k$, the radius of the subtree is $n-k$. We denote its partition function, restricted by fixing the root spin into up and down, by $Z^+(\vec{i}_m)$ and $Z^-(\vec{i}_m)$, respectively. The recursion relations for the partial partition functions of any two consecutive subtrees are expressed as

$$Z^{\pm}(\vec{i}_{m-1}) = y^{\pm 1} \prod_{\vec{i}_m}^{C(\vec{i}_{m-1})} [x^{\pm 1} Z^+(\vec{i}_m) + x^{\mp 1} Z^-(\vec{i}_m)], \tag{2}$$

where $x = \exp(\beta J)$, $y = \exp(\beta h)$, and $\beta = 1/T$, T being the temperature. Here $\Pi_{\vec{i}_m}^{C(\vec{i}_{m-1})}$ ($\Sigma_{\vec{i}_m}^{C(\vec{i}_{m-1})}$) means that the product (the sum) is over all vertices satisfying $\vec{i}_m \in C(\vec{i}_{m-1})$, where

$$C(\vec{a}) = \{\vec{i}_m | \vec{i}_m = (\vec{a}, i_m)\}. \tag{3}$$

The strategy by Stošić *et al.* is (i) finding the recursion relations for the field derivatives of the partition functions, (ii) taking the limit $h \to 0$, and (iii) then performing the iterations to reach the thermodynamic limit. The recursion relations for the field derivatives of the partition functions are

$$\begin{aligned} Z_h^{\pm}(\vec{i}_{m-1}) &= \pm y^{\pm 1} \prod_{\vec{i}_m}^{C(\vec{i}_{m-1})} [x^{\pm 1} Z^+(\vec{i}_m) + x^{\mp 1} Z^-(\vec{i}_m)] \\ &\quad + y^{\pm 1} \sum_{\vec{i}_m}^{C(\vec{i}_{m-1})} [x^{\pm 1} Z_h^+(\vec{i}_m) + x^{\mp 1} Z_h^-(\vec{i}_m)] \\ &\quad \times \left(\prod_{\vec{j}_m (\neq \vec{i}_m)}^{C(\vec{i}_{m-1})} [x^{\pm 1} Z^+(\vec{j}_m) + x^{\mp 1} Z^-(\vec{j}_m)] \right) \\ &= Z^{\pm}(\vec{i}_{m-1}) \left(\pm 1 + \sum_{\vec{i}_m}^{C(\vec{i}_{m-1})} \frac{x^{\pm 1} Z_h^+(\vec{i}_m) + x^{\mp 1} Z_h^-(\vec{i}_m)}{x^{\pm 1} Z^+(\vec{i}_m) + x^{\mp 1} Z^-(\vec{i}_m)} \right) \end{aligned} \tag{4}$$

and

$$\begin{aligned} Z_{hh}^{\pm}(\vec{i}_{m-1}) &= Z_h^{\pm}(\vec{i}_{m-1}) \left(\pm 1 + \sum_{\vec{i}_m}^{C(\vec{i}_{m-1})} \frac{x^{\pm 1} Z_h^+(\vec{i}_m) + x^{\mp 1} Z_h^-(\vec{i}_m)}{x^{\pm 1} Z^+(\vec{i}_m) + x^{\mp 1} Z^-(\vec{i}_m)} \right) \\ &\quad - Z^{\pm}(\vec{i}_{m-1}) \sum_{\vec{i}_m}^{C(\vec{i}_{m-1})} \left(\frac{x^{\pm 1} Z_h^+(\vec{i}_m) + x^{\mp 1} Z_h^-(\vec{i}_m)}{x^{\pm 1} Z^+(\vec{i}_m) + x^{\mp 1} Z^-(\vec{i}_m)} \right)^2 \\ &\quad + Z^{\pm}(\vec{i}_{m-1}) \sum_{\vec{i}_m}^{C(\vec{i}_{m-1})} \left(\frac{x^{\pm 1} Z_{hh}^+(\vec{i}_m) + x^{\mp 1} Z_{hh}^-(\vec{i}_m)}{x^{\pm 1} Z^+(\vec{i}_m) + x^{\mp 1} Z^-(\vec{i}_m)} \right), \end{aligned} \tag{5}$$

where

$$Z_h^{\pm}(\vec{i}_m) = \frac{\partial Z^{\pm}(\vec{i}_m)}{\partial(\beta h)}, \quad Z_{hh}^{\pm}(\vec{i}_m) = \frac{\partial^2 Z^{\pm}(\vec{i}_m)}{\partial(\beta h)^2}. \tag{6}$$

For a leaf spin, $Z^{\pm}(\vec{i}_n) = y^{\pm 1}$, $Z_h^{\pm}(\vec{i}_n) = \pm y^{\pm 1}$, and $Z_{hh}^{\pm}(\vec{i}_n) = y^{\pm 1}$. Thus, when we take the limit $h \to 0$, the following relations are satisfied at any radius n: $Z^+(\vec{i}_m) = Z^-(\vec{i}_m)$, $Z_h^+(\vec{i}_m) = -Z_h^-(\vec{i}_m)$, and $Z_{hh}^+(\vec{i}_m) = Z_{hh}^-(\vec{i}_m)$. For the zero-field case, it is sufficient to calculate the following recursion relations:

$$A(\vec{i}_{m-1}) = 1 + t \sum_{\vec{i}_m}^{C(\vec{i}_{m-1})} A(\vec{i}_m), \tag{7}$$

$$B(\vec{i}_{m-1}) = \sum_{\vec{i}_m}^{C(\vec{i}_{m-1})} B(\vec{i}_m) + (1 - t^2) \sum_{\vec{i}_m}^{C(\vec{i}_{m-1})} A(\vec{i}_m)^2, \tag{8}$$

and

TABLE I. Magnetization and susceptibility of the Ising model on tree like structures with radius $n\to\infty$.

Regular tree	Bethe-lattice-like structure Regular Bethe lattice [22]	Cayley-tree-like structure Regular Cayley tree [24]
$P(k)=\delta(k-z)$	$m=0$ for $T>T_c$ $m\neq 0$ for $T\leqslant T_c$ where $\tanh(J/T_c)=1/(z-1)$	$m=0$ at any $T(\neq 0)$ $\chi\to\infty$ below T_s where $\tanh^2(J/T_s)=1/(z-1)$
SF treelike network	SF Bethe lattice [8]	SF Cayley tree (this work)
$P(k)\propto k^{-\gamma}$	$\gamma>3$ $m=0$ for $T>T_c$ $m\neq 0$ for $T\leqslant T_c$ where $\tanh(J/T_c)=\frac{\langle k\rangle}{\langle k(k-1)\rangle}$ $\gamma\leqslant 3$ $m\neq 0$ at any finite T	$\gamma>4$ $m=0$ at any finite T $\chi\to\infty$ below T_s where $\tanh^2(J/T_s)=\frac{\langle k\rangle}{\langle k(k-1)\rangle}$ $\gamma\leqslant 4$ $m=0$ at any finite T $\chi\to\infty$ at any finite T

$$N(\vec{i}_{m-1}) = 1 + \sum_{\vec{i}_m}^{C(\vec{i}_{m-1})} N(\vec{i}_m), \tag{9}$$

where

$$A(\vec{i}_m) = \frac{Z_h^+(\vec{i}_m)}{Z^+(\vec{i}_m)}, \tag{10}$$

$$B(\vec{i}_m) = \frac{Z_{hh}^+(\vec{i}_m)}{Z^+(\vec{i}_m)} - \left(\frac{Z_h^+(\vec{i}_m)}{Z^+(\vec{i}_m)}\right)^2, \tag{11}$$

and $t=\tanh(\beta J)$. Here $N(\vec{i}_m)$ is the number of vertices of the subtree whose root vertex is $\vec{i}_m$. Finally we can obtain the total magnetization $m_n^\pm$ and the susceptibility χ_n of the tree as $m_n^\pm=\pm\langle A(\vec{i}_0)\rangle/\langle N(\vec{i}_0)\rangle$ and $\chi_n=\beta\langle B(\vec{i}_0)\rangle/\langle N(\vec{i}_0)\rangle$, respectively.

IV. RESULTS

Detailed calculations are given in the Appendix. From the recursion relations (7)–(9), $\langle A(\vec{i}_0)\rangle$, $\langle B(\vec{i}_0)\rangle$, and $\langle N(\vec{i}_0)\rangle$ are evaluated as

$$\langle A(\vec{i}_0)\rangle = 1 + ct\frac{\alpha^n t^n - 1}{\alpha t - 1}, \tag{12}$$

$$\begin{aligned}\frac{\langle B(\vec{i}_0)\rangle}{c(1-t^2)} &= \frac{\alpha^2t^2+\alpha t^2-\eta t^2-1}{(\alpha t-1)^2(\alpha t^2-1)}\frac{\alpha^n-1}{\alpha-1} \\ &+ \frac{2\alpha^{n-1}t(\eta t-\alpha^2 t-\alpha t+\alpha)}{(\alpha t-1)^2(t-1)}\frac{t^n-1}{t-1} \\ &+ \frac{(\eta-\alpha)\alpha^n t^2}{(\alpha t-1)^2(\alpha-1)}\frac{(\alpha t^2)^n-1}{\alpha t^2-1} \\ &+ \frac{(\alpha^2-\eta)(t+1)\alpha^{n-1}t^2}{(\alpha-1)(t-1)(\alpha t^2-1)}\frac{t^{2n}-1}{t^2-1},\end{aligned} \tag{13}$$

$$\langle N(\vec{i}_0)\rangle = 1 + c\frac{\alpha^n-1}{\alpha-1}, \tag{14}$$

where

$$c=\langle k\rangle, \quad \alpha=\frac{\langle k(k-1)\rangle}{\langle k\rangle}, \quad \eta=\frac{\langle k^2(k-1)\rangle}{\langle k\rangle}. \tag{15}$$

Consequently, the magnetization m_n^+ and the susceptibility χ_n of the Ising model on the SF Cayley tree with radius n have the following expressions:

$$m_n^+ = \left(1+ct\frac{\alpha^n t^n-1}{\alpha t-1}\right)\bigg/\left(1+c\frac{\alpha^n-1}{\alpha-1}\right), \tag{16}$$

$$\begin{aligned}\chi_n = \Bigg(&\frac{\alpha^2t^2+\alpha t^2-\eta t^2-1}{(\alpha t-1)^2(\alpha t^2-1)}\frac{\alpha^n-1}{\alpha-1} \\ &+ \frac{2\alpha^{n-1}t(\eta t-\alpha^2 t-\alpha t+\alpha)}{(\alpha t-1)^2(t-1)}\frac{t^n-1}{t-1} \\ &+ \frac{(\eta-\alpha)\alpha^n t^2}{(\alpha t-1)^2(\alpha-1)}\frac{(\alpha t^2)^n-1}{\alpha t^2-1} \\ &+ \frac{(\alpha^2-\eta)(t+1)\alpha^{n-1}t^2}{(\alpha-1)(t-1)(\alpha t^2-1)}\frac{t^{2n}-1}{t^2-1}\Bigg) \\ &\times\frac{c(1-t^2)}{k_BT}\bigg/\left(1+c\frac{\alpha^n-1}{\alpha-1}\right).\end{aligned} \tag{17}$$

These results indicate that interacting systems on SF networks with the two treelike structures show entirely different behaviors from each other in the thermodynamic limit (Table I).

In the limit $n\to\infty$, the magnetization of the SF Cayley tree becomes zero at any finite temperature in contrast to the SF Bethe lattice. The finite SF Cayley tree, however, has an extremely slow decay of magnetic ordering with increasing

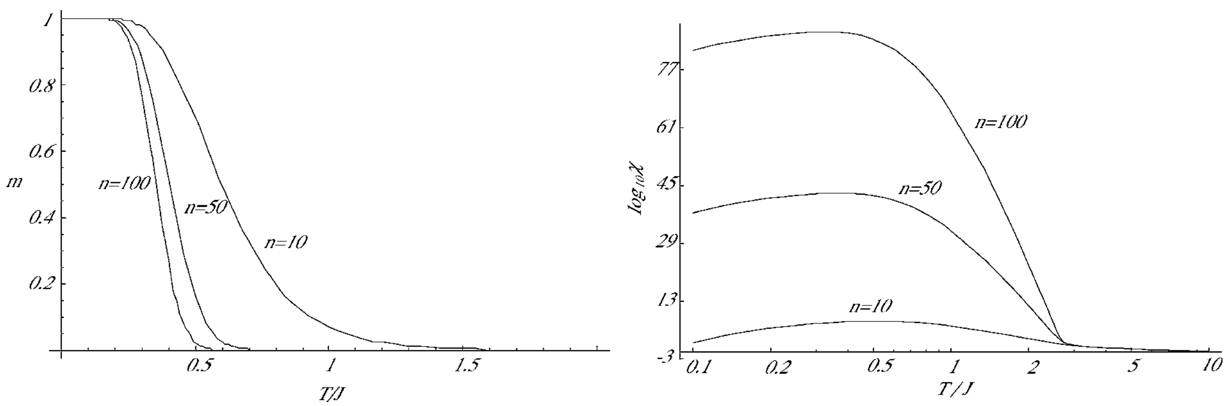

FIG. 2. Magnetization (a) and zero-field susceptibility (b) of the Ising model on a finite SF Cayley tree with radius $n=10$, 50, and 100. The degree distribution is $P(k)\propto k^{-\gamma}$ with $\gamma=2.7$. The maximum degree is 40.

system size, and has nonzero magnetization at low temperature even if at macroscopic size in the same way as the regular Cayley tree [26] (Fig. 2).

From Eq. (17), we find the temperature at which χ diverges: (i) for $\gamma>4$, χ diverges below T_s, where T_s is determined by $\tanh^2(J/T_s)=\langle k\rangle/\langle k(k-1)\rangle$, in a similar way as in the regular Cayley tree [27]; (ii) for $\gamma\leqslant 4$, χ diverges at any finite temperature due to the divergence of η.

Note that our expressions are not restricted within a power-law degree distribution. If we consider the case $P(k)=\delta(k-3)$, our expressions recover those for the regular Cayley tree [24]. On the other hand, by substituting $c=\alpha=\langle k\rangle$ and $\eta=\langle k^2\rangle$ into Eqs. (16) and (17), these expressions become the magnetization and the zero-field susceptibility of the Ising model on the Galton-Watson process having a given distribution $P(k)$.

V. SUMMARY

We have derived exact representations for the magnetization and zero-field susceptibility of the Ising model on a SF Cayley tree to show that the interacting system has quite different behavior depending on whether the networks are bounded by leaves or not. The Ising model on the SF Cayley tree has no magnetization at any finite temperature, and the susceptibility diverges below a certain temperature T_s depending on the exponent γ. The temperature T_s is given by $\tanh^2(J/T_s)=\langle k\rangle/\langle k(k-1)\rangle$ for $\gamma>4$, while T_s reaches infinity for $\gamma\leqslant 4$. Thus the susceptibility is divergent at any temperature for $\gamma\leqslant 4$.

In this paper, we replaced the vertices on the nth shell with leaves. We expect that the leaves affect the cooperative behavior on real SF networks, although there are not as many leaves as in our network model. In particular, some effects of leaves may appear in the dynamics of interacting systems. For the regular Cayley tree, it is known that the ferromagnetic Ising model has glassy behavior [28,29]. So far, it has become clear that hubs give surprising effects on networks while leaves have had less attention. Our results indicate the possibility that the competition of hubs and leaves leads to a variety of behaviors of interacting systems on SF networks.

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APPENDIX: THE CALCULATION OF THE RECURSION RELATIONS

First, we calculate $\langle A(\vec{i}_0)\rangle$ and $\langle N(\vec{i}_0)\rangle$ from Eqs. (7) and (9). Starting from the leaves, we obtain recursively $A(\vec{i}_n)=1$, $A(\vec{i}_{n-1})=1+t\Sigma_{\vec{i}_n}^{C(\vec{i}_{n-1})}1,\ldots$, and

$$A(\vec{i}_0) = 1 + t\sum_{\vec{i}_1}^{C(\vec{i}_0)} 1 + t^2\sum_{\vec{i}_1}^{C(\vec{i}_0)}\sum_{\vec{i}_2}^{C(\vec{i}_1)} 1 + \cdots + t^n\sum_{\vec{i}_1}^{C(\vec{i}_0)}\cdots\sum_{\vec{i}_n}^{C(\vec{i}_{n-1})} 1. \tag{A1}$$

Now we take the average. $\langle\Sigma_{\vec{i}_1}^{C(\vec{i}_0)}1\rangle$ is equal to the mean number of vertices on the first shell $\langle k\rangle\equiv c$, $\langle\Sigma_{\vec{i}_1}^{C(\vec{i}_0)}\Sigma_{\vec{i}_2}^{C(\vec{i}_1)}1\rangle$ is equal to the mean number of vertices on the second shell $[\langle k(k-1)\rangle/\langle k\rangle]\langle k\rangle\equiv\alpha c,\cdots$, and $\langle\Sigma_{\vec{i}_1}^{C(\vec{i}_0)}\cdots\Sigma_{\vec{i}_n}^{C(\vec{i}_{n-1})}1\rangle$ is the mean number of leaves $\alpha^{n-1}c$. Thus we obtain

$$\langle A(\vec{i}_0)\rangle = 1 + ct + c\alpha t^2 + \cdots + c\alpha^{n-1}t^n = 1 + \frac{\alpha^n t^n - 1}{\alpha t - 1}ct. \tag{A2}$$

Similarly, we find

$$\langle N(\vec{i}_0)\rangle = 1 + \frac{\alpha^n - 1}{\alpha - 1}c. \tag{A3}$$

Second, we calculate $\langle B(\vec{i}_0)\rangle$ from Eq. (8). Starting from the leaves, we obtain recursively $B(\vec{i}_n)=0$, $B(\vec{i}_{n-1})=(1-t^2)\Sigma_{\vec{i}_n}^{C(\vec{i}_{n-1})}A(\vec{i}_n)^2$, $B(\vec{i}_{n-2})=(1-t^2)\Sigma_{\vec{i}_{n-1}}^{C(\vec{i}_{n-2})}\Sigma_{\vec{i}_n}^{C(\vec{i}_{n-1})}A(\vec{i}_n)^2+(1-t^2)\Sigma_{\vec{i}_{n-1}}^{C(\vec{i}_{n-2})}A(\vec{i}_{n-1})^2,\ldots$, and

$$B(\vec{i}_0) = (1-t^2)\sum_{k=0}^{n-1}\sum_{\vec{i}_1}^{C(\vec{i}_0)}\cdots\sum_{\vec{i}_{k+1}}^{C(\vec{i}_k)} A(\vec{i}_{k+1})^2, \tag{A4}$$

where

$$A(\vec{i}_{k+1})^2 = \left(1 + t\sum_{\vec{i}_{k+2}}^{C(\vec{i}_{k+1})} 1 + \cdots + t^{n-k-1}\sum_{\vec{i}_{k+2}}^{C(\vec{i}_{k+1})}\cdots\sum_{\vec{i}_n}^{C(\vec{i}_{n-1})} 1\right)^2. \tag{A5}$$

We first calculate $\langle A(\vec{i}_{k+1})^2\rangle$. In calculation of the average of terms appearing in expanding the right-hand side of Eq. (A5), one should be careful in treating the average values such as $\langle \Sigma_{\vec{i}_{q+1}}^{C(\vec{i}_q)}\Sigma_{\vec{j}_{q+1}}^{C(\vec{j}_q)}1\rangle$. If $\vec{i}_q$ is equal to $\vec{j}_q$, this average reduces to $\Sigma[k^2(k-1)P(k)]/\langle k\rangle = [\langle k^2(k-1)\rangle]/\langle k\rangle \equiv \eta$, otherwise this average reduces to α^2. For example, one finds the average

$$\left\langle \sum_{\vec{i}_{k+2}}^{C(\vec{i}_{k+1})}\sum_{\vec{j}_{k+2}}^{C(\vec{j}_{k+1})}\cdots\sum_{\vec{i}_{k+p+1}}^{C(\vec{i}_{k+p})}\sum_{\vec{j}_{k+p+1}}^{C(\vec{j}_{k+p})} 1\right\rangle = \frac{1}{\alpha-1}[(\eta-\alpha)\alpha^{2p-1} + (\alpha^2-\eta)\alpha^{p-1}] \equiv f(p). \tag{A6}$$

Now we can express $\langle A(\vec{i}_{n-l})^2\rangle$ in terms of $f(p)$ as

$$\begin{aligned}\langle A(\vec{i}_{n-l})^2\rangle &= \left\langle\left(1 + t\sum_{\vec{i}_{n-l+1}}^{C(\vec{i}_{n-l})} 1 + \cdots + t^l\sum_{\vec{i}_{n-l+1}}^{C(\vec{i}_{n-l})}\cdots\sum_{\vec{i}_n}^{C(\vec{i}_{n-1})} 1\right)\left(1 + t\sum_{\vec{j}_{n-l+1}}^{C(\vec{j}_{n-l})} 1 + \cdots + t^l\sum_{\vec{j}_{n-l+1}}^{C(\vec{j}_{n-l})}\cdots\sum_{\vec{j}_n}^{C(\vec{j}_{n-1})} 1\right)\right\rangle \\ &= 1 + t\alpha + t^2\alpha^2 + t^3\alpha^3 + \cdots + t^l\alpha^l + t\alpha + t^2 f(1) + t^3 f(1)\alpha + t^4 f(1)\alpha^2 + \cdots + t^{l+1}f(1)\alpha^{l-1} + \cdots + t^l\alpha^l + t^{l+1}f(1)\alpha^{l-1} \\ &\quad + t^{l+2}f(2)\alpha^{l-2} + \cdots + t^{2l}f(l) = \frac{\alpha^2t^2 + \alpha t^2 - \eta t^2 - 1}{(\alpha t-1)^2(\alpha t^2-1)} + \frac{2t(\eta t - \alpha^2 t - \alpha t + \alpha)}{(\alpha t-1)^2(t-1)}(\alpha t)^l + \frac{(\eta-\alpha)\alpha t^2}{(\alpha t-1)^2(\alpha-1)}(\alpha^2t^2)^l \\ &\quad + \frac{(\alpha^2-\eta)(t+1)t^2}{(\alpha-1)(t-1)(\alpha t^2-1)}(\alpha t^2)^l.\end{aligned} \tag{A7}$$

Last, we consider the average of $B(\vec{i}_0)$,

$$\begin{aligned}\langle B(\vec{i}_0)\rangle &= (1-t^2)\sum_{k=0}^{n-1}\left\langle\sum_{\vec{i}_1}^{C(\vec{i}_0)}\cdots\sum_{\vec{i}_{k+1}}^{C(\vec{i}_k)}\right\rangle\langle A(\vec{i}_{k+1})^2\rangle \\ &= (1-t^2)c\sum_{k=0}^{n-1}\alpha^k\langle A(\vec{i}_{k+1})^2\rangle.\end{aligned} \tag{A8}$$

Thus substituting Eq. (A7) into Eq. (A8), we get the final expression of $\langle B_n(\vec{i}_0)\rangle$, i.e., Eq. (13).

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